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The Twelve Surfaces of Darboux and the Transformation of Moutard.

BY LUTHER PFAHLER EISENHART.

It is a well-known fact that the problem of the infinitesimal deformation of a surface S is equivalent to the determination of the surfaces S_1, each of which is in one-to-one correspondence with S in such a way that corresponding linear elements are orthogonal. The problem is equivalent also to the determination of the surfaces S_0, each of which corresponds to S in such a way that the tangent planes at corresponding points are parallel and the asymptotic lines on either surface correspond to a conjugate system on the other. Darboux in the fourth volume of his *Leçons** observes that the relation between S and S_1 being reciprocal, the surface S determines an infinitesimal deformation of S_1 and there is a surface S_3 associated thereby with S_1 which is analogous to the corresponding surface S_0 associated with S. In like manner the deformation of S_0 determined by S introduces a surface S_2 which corresponds to S_0 with orthogonality of linear elements. This process can be continued in both ways, and the coordinates of each new surface can be found directly. Darboux has derived the expressions for these coordinates, and has observed that the system is closed, consisting of twelve surfaces.

In the present paper we derive the same results in a similar manner, but we are concerned primarily with the determination of the reason for the closure. One of the fundamental reasons is expressed in THEOREM III and the other is that the transformation by reciprocal polars with respect to the imaginary sphere, $x^2 + y^2 + z^2 + 1 = 0$, transforms a W-congruence into a congruence of the same kind. These results appear in §§ 1–4, and in § 5 it is shown that the twelve surfaces are distinct.

In § 6 we recall a theorem of Bianchi, namely that if S is a focal sheet of each of two W-congruences for which the other focal sheets are denoted by S_5

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* pp. 48–72.

and S_{52} respectively, each of the latter surfaces is a focal sheet of another congruence, which two congruences are such that the other focal sheets are one and the same surface S'. With each of the pairs of surfaces (S, S_5), (S, S_{52}), (S', S_5), (S', S_{52}) there may be associated ten other surfaces after the manner of § 3, giving in all forty surfaces. We consider in particular the twenty surfaces consisting of the set S_0, S, S_1, S_3, S_5, S_7 and the analogous surfaces for each other of the above pairs. In § 7 it is shown that these twenty surfaces are distinct provided that S_0 and the similar surface S_{02} of the pair (S, S_{52}) are not homothetic, and provided that the conjugate system on S_0 corresponding to the asymptotic lines on S does not have equal point invariants. This exceptional case is investigated in §§ 8–10, and it is found incidentally that the problem involved has to do with the determination of isothermal-conjugate systems of lines on surfaces.

1. Let ξ, η, ζ be three particular solutions of an equation of the form

$$\frac{\partial^2 \theta}{\partial u \, \partial v} = M\theta, \tag{1}$$

where M is a function of u and v. Upon the surface S whose coordinates x, y, z are defined by quadratures of the form

$$\frac{\partial x}{\partial u} = -\begin{vmatrix} \eta & \zeta \\ \frac{\partial \eta}{\partial u} & \frac{\partial \zeta}{\partial u} \end{vmatrix}, \quad \frac{\partial x}{\partial v} = \begin{vmatrix} \eta & \zeta \\ \frac{\partial \eta}{\partial v} & \frac{\partial \zeta}{\partial v} \end{vmatrix} \tag{2}$$

the parametric curves are the asymptotic lines.*

Let σ be any particular solution of equation (1). This equation may accordingly be written

$$\frac{\partial^2 \theta}{\partial u \, \partial v} = \frac{1}{\sigma} \frac{\partial^2 \sigma}{\partial u \, \partial v} \theta. \tag{3}$$

The surface S_1, whose coordinates x_1, y_1, z_1 are given by the quadratures

$$\frac{\partial x_1}{\partial u} = -\sigma^2 \frac{\partial}{\partial u}\left(\frac{\xi}{\sigma}\right), \quad \frac{\partial x_1}{\partial v} = \sigma^2 \frac{\partial}{\partial v}\left(\frac{\xi}{\sigma}\right), \tag{4}$$

and similar equations in y_1 and z_1, corresponds to S with orthogonality of linear elements; that is,

$$dx \, dx_1 + dy \, dy_1 + dz \, dz_1 = 0. \tag{5}$$

Moreover, the parametric curves on S_1 form a conjugate system with equal point invariants. In fact, x_1, y_1, z_1 are solutions of the equation

$$\frac{\partial^2 \phi}{\partial u \, \partial v} = \frac{\partial \log \sigma}{\partial v} \frac{\partial \phi}{\partial u} + \frac{\partial \log \sigma}{\partial u} \frac{\partial \phi}{\partial v}. \tag{6}$$

For the sake of brevity we shall say that S_1 is an *ortho-surface* of S.

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* Bianchi, *Lezioni*, Vol. I, p. 163; German translation, p. 132; also, Eisenhart, *Differential Geometry*, p. 193.

We have seen[*] that equation (5) may be replaced by the three equations

$$\left.\begin{aligned} dx_1 &= y_0\,dz - z_0\,dy, \\ dy_1 &= z_0\,dx - x_0\,dz, \\ dz_1 &= x_0\,dy - y_0\,dx, \end{aligned}\right\} \tag{7}$$

and that the surface S_0 whose coordinates are x_0, y_0, z_0 is the surface associate to S in the infinitesimal deformation of the latter determined by S_1. We shall express the relation between the three surfaces S, S_1, S_0 by saying that S_0 is *the associate of S corresponding to S_1*, and that S_1 is *the ortho-surface of S corresponding to S_0*.

Since S and S_0 correspond with parallelism of tangent planes, the parametric curves on S_0 form a conjugate system with equal tangential invariants.

When the values from (2) and (4) are substituted in equations (7), we obtain

$$\left.\begin{gathered} x_0 F(\xi, \eta, \zeta) = -F(\sigma, \eta, \zeta), \quad y_0 F(\xi, \eta, \zeta) = -F(\xi, \sigma, \zeta), \\ z_0 F(\xi, \eta, \zeta) = -F(\xi, \eta, \sigma), \end{gathered}\right\} \tag{8}$$

where the function $F(\theta_1, \theta_2, \theta_3)$ is defined by

$$F(\theta_1, \theta_2, \theta_3) = \begin{vmatrix} \theta_1 & \theta_2 & \theta_3 \\ \dfrac{\partial\theta_1}{\partial u} & \dfrac{\partial\theta_2}{\partial u} & \dfrac{\partial\theta_3}{\partial u} \\ \dfrac{\partial\theta_1}{\partial v} & \dfrac{\partial\theta_2}{\partial v} & \dfrac{\partial\theta_3}{\partial v} \end{vmatrix}. \tag{9}$$

Since the relation between S and S_1 is perfectly reciprocal, there exists a unique surface S_3 which is associate to S_1 corresponding to S. Its coordinates are given by equations of the form

$$dx = y_3\,dz_1 - z_3\,dy_1. \tag{10}$$

By means of (2) and (4) we find that

$$x_3 = \frac{\xi}{\sigma}, \quad y_3 = \frac{\eta}{\sigma}, \quad z_3 = \frac{\zeta}{\sigma}. \tag{11}$$

One consequence of this result is the theorem:

THEOREM I. *The radius vector of an associate surface of a surface S is parallel to the normal to the corresponding ortho-surface.*

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[*] *American Journal*, Vol. XXIV (1902), p. 186; *Differential Geometry*, p. 382.

In consequence of the fundamental identity

$$\sigma F(\xi, \eta, \zeta) = \xi F(\sigma, \eta, \zeta) + \eta F(\xi, \sigma, \zeta) + \zeta F(\xi, \eta, \sigma), \tag{12}$$

from (8) and (11) follows the relation

$$x_0 x_3 + y_0 y_3 + z_0 z_3 + 1 = 0. \tag{13}$$

From (7) we have also

$$\Sigma x_0 \frac{\partial x_1}{\partial u} = 0, \quad \Sigma x_0 \frac{\partial x_1}{\partial v} = 0, \tag{14}$$

which in consequence of (4) and (11) may be written

$$\Sigma x_0 \frac{\partial x_3}{\partial u} = 0, \quad \Sigma x_0 \frac{\partial x_3}{\partial v} = 0.$$

Hence we have

THEOREM II. *The surfaces S_0 and S_3 are polar reciprocals of one another with respect to the imaginary sphere*

$$x^2 + y^2 + z^2 + 1 = 0. \tag{15}$$

2. Since the relation between associate surfaces is reciprocal, there is a surface S_5 which is the ortho-surface of S_3 corresponding to S_1. Its coordinates x_5, y_5, z_5 are given by equations of the form

$$dx_5 = y_1 dz_3 - z_1 dy_3. \tag{16}$$

If these equations be subtracted from the corresponding equations (10), and the resulting equations be integrated, we have

$$\left.\begin{aligned} x_5 - x &= y_1 z_3 - z_1 y_3, \\ y_5 - y &= z_1 x_3 - x_1 z_3, \\ z_5 - z &= x_1 y_3 - y_1 x_3; \end{aligned}\right\} \tag{17}$$

in this case we have taken the additive constants equal to zero, thus assuming a particular position in space for S_5.

If we put

$$\xi_1 = \frac{x_1}{\sigma}, \quad \eta_1 = \frac{y_1}{\sigma}, \quad \zeta_1 = \frac{z_1}{\sigma}, \tag{18}$$

then by THEOREM I the quantities ξ_1, η_1, ζ_1 are proportional to the direction-cosines of the normal to S_5.

By means of (11) and (18) equations (17) are reducible to

$$\left.\begin{aligned} x_5 - x &= \eta_1 \zeta - \zeta_1 \eta, \\ y_5 - y &= \zeta_1 \xi - \xi_1 \zeta, \\ z_5 - z &= \xi_1 \eta - \eta_1 \xi, \end{aligned}\right\} \tag{19}$$

and equations (4) become

$$\frac{\partial}{\partial u}\left(\sigma\xi_1\right) = -\sigma^2\frac{\partial}{\partial u}\left(\frac{\xi}{\sigma}\right), \quad \frac{\partial}{\partial v}\left(\sigma\xi_1\right) = \sigma^2\frac{\partial}{\partial v}\left(\frac{\xi}{\sigma}\right). \tag{20}$$

From these equations we find that ξ_1, η_1, ζ_1 are solutions of the equation

$$\frac{\partial^2\theta_1}{\partial u\,\partial v} = \sigma\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma}\right).\,\theta. \tag{21}$$

Moreover, from (19) it follows that

$$\frac{\partial x_5}{\partial u} = -\begin{vmatrix} \eta_1 & \zeta_1 \\ \frac{\partial\eta_1}{\partial u} & \frac{\partial\zeta_1}{\partial u} \end{vmatrix}, \quad \frac{\partial x_5}{\partial v} = \begin{vmatrix} \eta_1 & \zeta_1 \\ \frac{\partial\eta_1}{\partial v} & \frac{\partial\zeta_1}{\partial v} \end{vmatrix}. \tag{22}$$

Hence the parametric curves on S_5 are its asymptotic lines.

From (19) it is seen that

$$\Sigma\,\xi\,(x_5 - x) = 0, \quad \Sigma\,\xi_1(x_5 - x) = 0.$$

Hence the line joining corresponding points on S and S_5 is tangent to both surfaces, and as the asymptotic lines on these surfaces correspond these joins form a *W*-congruence.

For convenience we represent the relations between the surfaces S, S_0, S_1, S_3, S_5 as in Fig. 1, where a vertical join indicates that the joined surfaces are associate to one another, and a horizontal line joins a surface and an ortho-surface.

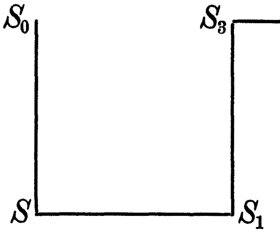

FIG. 1.

We shall prove the following converse result:

THEOREM III. *Let S and S_5 be the focal sheets of a W-congruence; if S_1 is an ortho-surface of S and S_3 is the corresponding associate surface of S_1, and also if S_3 is an ortho-surface of S_5, then S_1 is the associate of S_3 corresponding to S_5.*

Bianchi has shown* that if S and S_5 are the focal sheets of a *W*-congruence referred to the asymptotic lines of these surfaces, they may be defined by equations of the form (2) and (22) with the relations (19) and (20) between the quantities involved.

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* Vol. II, pp. 51–56; German translation, pp. 316–319; also, Eisenhart, *Differential Geometry*, pp. 417–420.

Since S_3 is an ortho-surface of S_5, it follows from (4) that

$$\frac{\partial x_3}{\partial u} = -\sigma_1^2 \frac{\partial}{\partial u}\left(\frac{\xi_1}{\sigma_1}\right), \quad \frac{\partial x_3}{\partial v} = \sigma_1^2 \frac{\partial}{\partial v}\left(\frac{\xi_1}{\sigma_1}\right), \tag{23}$$

where σ_1 is a solution of equation (21). But from the definition of S_3, its coordinates are of the form (11). When this value is substituted in the first of equations (23), we obtain

$$\left(\frac{1}{\sigma} - \sigma_1\right)\frac{\partial \xi_1}{\partial u} + \left(\frac{1}{\sigma^2}\frac{\partial \sigma}{\partial u} + \frac{\partial \sigma_1}{\partial u}\right)\xi_1 = 0.$$

Since this equation holds also for η_1 and ζ_1, we must have $\sigma_1 = 1/\sigma$, and consequently the theorem is proved.

3. The process by which we obtained the surfaces S_3 and S_5 may be continued. Thus there is a surface S_7 which is the associate to S_5 corresponding to S_3. In turn we obtain surfaces S_9 and S_{11}, represented in Fig. 2, where the lines have the same significance as in Fig. 1.

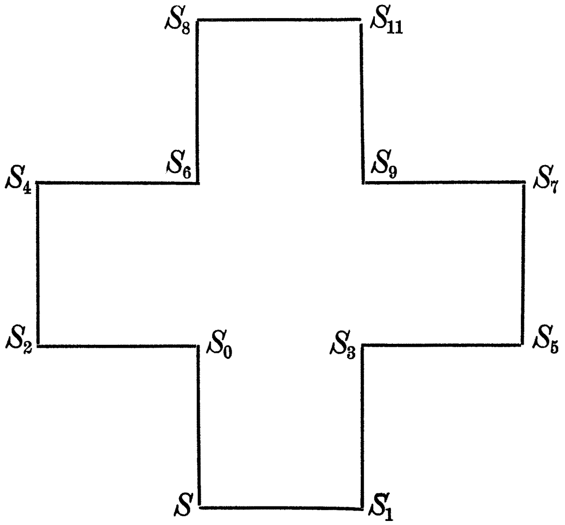

FIG. 2.

In order to obtain the coordinates x_7, y_7, z_7 of S_7, we observe that by THEOREM I the radius vector of S_7 is parallel to the normal to S_3, consequently to the normal to S_1, and therefore it is parallel to the radius vector of S_0. Combining this result with the fact that S_7 and S_1 are polar reciprocal with respect to the sphere (15), we have

$$\frac{x_7}{x_0} = \frac{y_7}{y_0} = \frac{z_7}{z_0} = -\frac{1}{\Sigma\, x_0 x_1}. \tag{24}$$

By reasoning of the same sort, we obtain by means of (18) the coordinates of S_{11}. They are given by

$$\frac{x_{11}}{\xi_1} = \frac{y_{11}}{\eta_1} = \frac{z_{11}}{\zeta_1} = -\frac{1}{\Sigma\,\xi_1 x_5} = -\frac{1}{\omega_1}, \tag{25}$$

where

$$\omega_1 = \Sigma\,\xi_1 x. \tag{26}$$

Since S_3 and S_9 are the focal sheets of a *W*-congruence, we have equations analogous to (17), namely

$$x_9 - x_3 = y_5 z_7 - z_5 y_7,$$

and similar expressions for y_9 and z_9. In consequence of (11), (19), (24), the above equation is reducible to

$$x_9 = \frac{\xi}{\sigma} - \frac{(y + \zeta_1 \xi - \zeta \xi_1) z_0 - (z + \xi_1 \eta - \xi \eta_1) y_0}{\Sigma\, x_0 x_1}. \tag{27}$$

4. In a similar manner we may proceed from S in the other direction and obtain a suite of surfaces S_0, S_2, ..., represented in Fig. 2. It is our purpose to show that S_8 and S_{11} are ortho-surfaces of one another corresponding to S_6 and S_9, and consequently that there is a closed cycle of twelve surfaces, and thus to justify the horizontal join of S_8 and S_{11} in Fig. 2.

The surfaces S_1 and S_2 are the focal sheets of a *W*-congruence, and consequently we have

$$x_2 = x_1 + y z_0 - z y_0, \tag{28}$$

and similar expressions for y_2 and z_2.

It can likewise be shown that the coordinates of S_8 are given by

$$\frac{x_8}{x} = \frac{y_8}{y} = \frac{z_8}{z} = -\frac{1}{\Sigma x x_1} = -\frac{1}{\sigma \omega_1}. \tag{29}$$

The surfaces S_8 and S_7 are polar reciprocals of S_2 and S_1 respectively with respect to the sphere (15). Since the relation of polar reciprocals with respect to this sphere is a contact transformation which preserves asymptotic lines, the surfaces S_8 and S_7 are the focal sheets of a *W*-congruence. Hence the coordinates of these surfaces are related by equations analogous to (17). These equations are in fact

$$x_8 - x_7 = y_9 z_{11} - z_9 y_{11}, \tag{30}$$

and similarly for y and z; for, the equation (30) is satisfied identically by the values from (24), (25), (27) and (29). Hence S_8 is the ortho-surface of S_{11} corresponding to S_9.

Since S_8 and S_{11} are ortho-surfaces, and since S_4 and S_{11} are the focal sheets of a W-congruence, being the polar reciprocals of S and S_5 respectively, it follows from THEOREM III that S_{11} is the ortho-surface of S_8 corresponding to S_6. Hence we have the following theorem of Darboux:

THEOREM IV. *Let S be any surface, S_1 an ortho-surface, and S_0 the corresponding associate surface; nine other surfaces can be found directly which form with these three a closed cycle of twelve surfaces, such that of any three contiguous surfaces of the cycle, S_a, S_b, S_c, either S_a is an ortho-surface of S_b and S_c the corresponding associate surface, or vice-versa.*

5. We shall determine whether the twelve surfaces are distinct. To this end we remark that as the asymptotic lines are parametric on S, S_5, S_4 and S_{11}, if S coincides with any of the twelve surfaces it must be one of the latter three.

The surface S cannot correspond with S_4 unless it be the sphere (15).* From (17) it follows that if S and S_5 coincide

$$\frac{x_3}{x_1} = \frac{y_3}{y_1} = \frac{z_3}{z_1}.$$

Since S_1 and S_3 are associate surfaces referred to their common conjugate system, we must† have

$$\frac{\partial x_3}{\partial u} = \lambda \frac{\partial x_1}{\partial u}, \quad \frac{\partial x_3}{\partial v} = -\lambda \frac{\partial x_1}{\partial v},$$

and similar equations in y and z, where λ is in general a function of u and v. Since these sets of equations are evidently inconsistent, the surfaces S and S_5 are always distinct.

It remains for us to consider the possible coincidence of S and S_{11}. From (25) and (18) it follows that if S and S_{11} coincide, we have

$$x_1 = tx, \quad y_1 = ty, \quad z_1 = tz,$$

where t is a factor of proportionality. Suppose that S and S_1 are referred to the system of lines which is orthogonal for both surfaces. In this case

$$\Sigma \frac{\partial x_1}{\partial u}\frac{\partial x_1}{\partial v} = 0, \quad \Sigma \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} = 0.$$

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* It should be remarked that if S is the unit sphere, S_4 is the same sphere, but diametrically opposite points correspond (cf. § 10).

† Eisenhart, *Differential Geometry*, p. 380.

If we substitute the above values in the first of these equations and reduce the result by means of the second, we obtain

$$t\frac{\partial t}{\partial u}\Sigma x\frac{\partial x}{\partial v}+t\frac{\partial t}{\partial v}\Sigma x\frac{\partial x}{\partial u}+\frac{\partial t}{\partial u}\frac{\partial t}{\partial v}\Sigma x^2=0.$$

From (5) it follows that

$$\Sigma\frac{\partial x}{\partial u}\frac{\partial x_1}{\partial v}+\Sigma\frac{\partial x}{\partial v}\frac{\partial x_1}{\partial u}=0.$$

When the above values of x_1, y_1, z_1 are substituted in this equation, we obtain

$$\frac{\partial t}{\partial v}\Sigma x\frac{\partial x}{\partial u}+\frac{\partial t}{\partial u}\Sigma x\frac{\partial x}{\partial v}=0.$$

Comparing these two equations for the determination of t, we see that t must be a constant. But in this case equation (5) reduces to $\Sigma dx^2=0$, which evidently is impossible. Hence S does not coincide with any other of the twelve surfaces. Moreover, since any surface of the cycle can be taken as the initial surface, it follows that all of the twelve surfaces are distinct.

6. Bianchi* has shown that if σ_1 and σ_2 are particular solutions of equation (1), the functions σ_1' and σ_2', given by

$$\frac{\partial}{\partial u}\left(\sigma_1\sigma_1'\right)=-\sigma_1^2\frac{\partial}{\partial u}\left(\frac{\sigma_2}{\sigma_1}\right),\quad \frac{\partial}{\partial v}\left(\sigma_1\sigma_1'\right)=\sigma_1^2\frac{\partial}{\partial v}\left(\frac{\sigma_2}{\sigma_1}\right),\tag{31}$$

$$\frac{\partial}{\partial u}\left(\sigma_2\sigma_2'\right)=-\sigma_2^2\frac{\partial}{\partial u}\left(\frac{\sigma_1}{\sigma_2}\right),\quad \frac{\partial}{\partial v}\left(\sigma_2\sigma_2'\right)=\sigma_2^2\frac{\partial}{\partial v}\left(\frac{\sigma_1}{\sigma_2}\right),\tag{32}$$

satisfy the conditions

$$\sigma_2\sigma_2'=-\sigma_1\sigma_1',\quad \sigma_1'\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma_1'}\right)=\sigma_2'\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma_2'}\right),\tag{33}$$

and are solutions of the respective equations

$$\frac{\partial^2\theta_1}{\partial u\,\partial v}=\sigma_1\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma_1}\right)\theta_1,\tag{34}$$

$$\frac{\partial^2\theta_2}{\partial u\,\partial v}=\sigma_2\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma_2}\right)\theta_2.\tag{35}$$

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* Vol. II, p. 70.

Let ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2 be the solutions of equations (34) and (35) respectively, given by quadratures of the form (20). It is readily shown that the functions ξ', η', ζ', defined by

$$\xi' = \xi + t(\xi_1 - \xi_2), \quad \eta' = \eta + t(\eta_1 - \eta_2), \quad \zeta' = \zeta + t(\zeta_1 - \zeta_2), \tag{36}$$

where

$$t = -\frac{\sigma_2}{\sigma_1'} = \frac{\sigma_1}{\sigma_2'}, \tag{37}$$

are solutions of the equation

$$\frac{\partial^2 \theta'}{\partial u\, \partial v} = \sigma_1' \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sigma_1'}\right)\theta' = \sigma_2' \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sigma_2'}\right)\theta'. \tag{38}$$

The functions ξ_2, η_2, ζ_2 determine a surface S_{52} whose coordinates are given by equations analogous to (22). The surfaces S and S_{52} are the focal sheets of a W-congruence, and their coordinates are in the relation

$$x_{52} = x + (\eta_2 \zeta - \zeta_2 \eta), \quad y_{52} = y + (\zeta_2 \xi - \xi_2 \zeta), \quad z_{52} = z + (\xi_2 \eta - \eta_2 \xi). \tag{39}$$

Furthermore, the surface S', defined by

$$\left.\begin{aligned} x' &= x + t(\eta_1 \zeta_2 - \zeta_1 \eta_2), \\ y' &= y + t(\zeta_1 \xi_2 - \xi_1 \zeta_2), \\ z' &= z + t(\xi_1 \eta_2 - \eta_1 \xi_2), \end{aligned}\right\} \tag{40}$$

is such that S_5 and S' are focal sheets of a W-congruence, and likewise S_{52} and S'. For convenience we shall say that the four surfaces S, S_5, S_{52}, S' form a *quatern*. The asymptotic lines correspond on all four surfaces, and the latter are determined by quadratures when two integrals of equation (1) are known.

With each of the three W-congruences thus introduced there are associated four surfaces analogous to S_0, S_1, S_3, S_7. They are denoted as follows:

$$\left.\begin{matrix} S_0, & S, & S_1, & S_3, & S_5, & S_7; \\ S_{02}, & S, & S_{12}, & S_{32}, & S_{52}, & S_{72}; \\ S_0', & S', & S_1', & S_3', & S_5, & S_7'; \\ S_{02}', & S', & S_{12}', & S_{32}', & S_{52}, & S_{72}'. \end{matrix}\right\} \tag{41}$$

It is desirable that we should know under what conditions all of these surfaces are distinct.

7. Since the parametric curves on the surfaces of the second and fifth columns of (41) are the asymptotic lines on these surfaces, and since this is true of none of the other surfaces, it follows that these four surfaces are distinct from all the others.

In a similar manner, the surfaces of the first and last columns are characterized by the fact that the parametric curves form a conjugate system with equal tangential invariants, and in general the surfaces of the third and fourth columns do not possess this property. But two of these surfaces cannot coincide, unless the same is true of two of the surfaces S, S_5, S_{52}, S'.

From (11) and (18) it follows that the coordinates of the surfaces of the third and fourth columns are of the form

$$\left.\begin{array}{llll} x_1 = \xi_1 \sigma_1, & x_{12} = \xi_2 \sigma_2, & x'_3 = \xi' \sigma'_1, & x'_{32} = \xi' \sigma'_2, \\ x_3 = \dfrac{\xi}{\sigma_1}, & x_{32} = \dfrac{\xi}{\sigma_2}, & x'_1 = \dfrac{\xi_1}{\sigma'_1}, & x'_{12} = \dfrac{\xi_2}{\sigma'_2}. \end{array}\right\} \tag{42}$$

The coincidence of S_1 with any of these surfaces other than S'_1 necessitates the coincidence of two of the surfaces S, S_5, S_{52}, S'.

We consider the case where S_1 and S'_1 are coincident. From (42) it follows that we must have $\sigma_1 \sigma'_1 = 1$. Now equations (31), (32) reduce to

$$\sigma_2 = \varkappa \sigma_1, \tag{43}$$

where $\varkappa$ denotes a constant. Hence equations (34) and (35) are equivalent. From (20) we find that

$$\frac{\partial}{\partial u}\left(\sigma_1 \xi_2\right) = \frac{\partial}{\partial u}\left(\sigma_1 \xi_1\right), \quad \frac{\partial}{\partial v}\left(\sigma_1 \xi_2\right) = \frac{\partial}{\partial v}\left(\sigma_1 \xi_1\right),$$

whence we have

$$\xi_2 = \xi_1 + \frac{a}{\sigma_1}, \quad \eta_2 = \eta_1 + \frac{b}{\sigma_1}, \quad \zeta_2 = \zeta_1 + \frac{c}{\sigma_1}, \tag{44}$$

a, b, c being constants.

Since $\sigma_1 \sigma'_1 = 1$, equation (38) is equivalent to (1), and equations (36) reduce to

$$\xi' = \xi + \varkappa a \sigma_1, \quad \eta' = \eta + \varkappa b \sigma_1, \quad \zeta' = \zeta + \varkappa c \sigma_1. \tag{45}$$

From (8) and analogous formulas we find that S_7 and S'_7 coincide, that the surfaces of each pair S_0, S_{02}; S_{72}, S'_{72}; S'_0, S'_{02} are homothetic transforms of one another, and that

$$\frac{x_{72}}{x_7} = \frac{y_{72}}{y_7} = \frac{z_{72}}{z_7} = \frac{1}{\varkappa\left(1 - a x_7 - b y_7 - c z_7\right)},$$

$$\frac{x'_0}{x_0} = \frac{y'_0}{y_0} = \frac{z'_0}{z_0} = \frac{1}{1 - \varkappa\left(a x_0 + b y_0 + c z_0\right)}.$$

By means of (42), (43), (44), (45) it can be shown that the surfaces S_{12}, S'_{12} can be obtained from S_1 by homothetic transformations and translations; and similarly S_{32}, S'_3, S''_{32} from S_3.

The foregoing results may be stated thus:

THEOREM V. *When* σ_1 *and* σ_2 *are not in constant ratio, the twenty surfaces* (41) *are distinct, provided that the parametric curves on any of the surfaces in the first or last column do not form a conjugate system with equal point invariants.*

8. Thus far we have excluded the case where the parametric conjugate system on S_0 has equal point invariants as well as equal tangential invariants; that is, where

$$\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}_0 = \frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix}_0, \tag{46}$$

the symbols $\begin{Bmatrix}rs\\t\end{Bmatrix}_0$ being formed with respect to the linear element of S_0. We consider this case.

The Codazzi equations* for S_0 are of the form

$$\left.\begin{aligned}\frac{\partial \log D_0}{\partial v} &= \begin{Bmatrix}12\\1\end{Bmatrix}' - \frac{D''_0}{D_0}\begin{Bmatrix}11\\2\end{Bmatrix}',\\ \frac{\partial \log D''_0}{\partial u} &= \begin{Bmatrix}12\\2\end{Bmatrix}' - \frac{D_0}{D''_0}\begin{Bmatrix}22\\1\end{Bmatrix}',\end{aligned}\right\} \tag{47}$$

the symbols $\begin{Bmatrix}rs\\t\end{Bmatrix}'$ being formed with respect to the linear element of the spherical representation of S and S_0, namely

$$d\sigma^2 = \mathcal{E}du^2 + 2\mathcal{F}du\,dv + \mathcal{G}dv^2. \tag{48}$$

In consequence of the following relations:†

$$\frac{D''_0}{D_0}\begin{Bmatrix}11\\2\end{Bmatrix}' = -\begin{Bmatrix}12\\1\end{Bmatrix}_0, \quad \frac{D_0}{D''_0}\begin{Bmatrix}22\\1\end{Bmatrix}' = -\begin{Bmatrix}12\\2\end{Bmatrix}_0, \tag{49}$$

equations (47) are reducible to

$$\left.\begin{aligned}\frac{\partial \log D_0}{\partial v} &= \begin{Bmatrix}12\\1\end{Bmatrix}' + \begin{Bmatrix}12\\1\end{Bmatrix}_0,\\ \frac{\partial \log D''_0}{\partial u} &= \begin{Bmatrix}12\\2\end{Bmatrix}' + \begin{Bmatrix}12\\2\end{Bmatrix}_0.\end{aligned}\right\} \tag{47'}$$

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* Bianchi, I, 166; German translation, p. 134; Eisenhart, p. 161.

† Bianchi, I, 167; German translation, p. 135; Eisenhart, p. 201.

In consequence of (46) and the fact that the asymptotic lines on S are parametric. we have from (47′)

$$\frac{\partial^2}{\partial u\,\partial v}\log\frac{D_0}{D_0''}=0.$$

Hence the parametric system on S_0 is isothermal-conjugate. As the parameters of the asymptotic lines on S were not of any particular kind, the parameters may be chosen so that $D_0=\varepsilon D_0''$, where ε is ± 1 according as the curvature of S_0 is positive or negative.

In this case equations (47) are reducible to

$$\left.\begin{aligned}\frac{\partial\log D_0}{\partial v}&=\left\{\begin{matrix}12\\1\end{matrix}\right\}'-\varepsilon\left\{\begin{matrix}11\\2\end{matrix}\right\}',\\ \frac{\partial\log D_0}{\partial u}&=\left\{\begin{matrix}12\\2\end{matrix}\right\}'-\varepsilon\left\{\begin{matrix}22\\1\end{matrix}\right\}'.\end{aligned}\right\}\tag{50}$$

In order that these equations be consistent, we must have

$$\frac{\partial}{\partial u}\left\{\begin{matrix}11\\2\end{matrix}\right\}'=\frac{\partial}{\partial v}\left\{\begin{matrix}22\\1\end{matrix}\right\}'.\tag{51}$$

When this condition is satisfied, two sets of solutions of (47) follow from (50) by quadratures. One set is such that $D_0=D_0''$, and the other that $D_0=-D_0''$. The further determination of the surfaces determined by these values requires quadratures of the form*

$$\left.\begin{aligned}\frac{\partial x_0}{\partial u}&=\frac{D_0}{\sqrt{\mathcal{E}\mathcal{G}-\mathcal{F}^2}}\left(-\mathcal{G}\frac{\partial X}{\partial u}+\mathcal{F}\frac{\partial X}{\partial v}\right),\\ \frac{\partial x_0}{\partial v}&=\frac{D_0''}{\sqrt{\mathcal{E}\mathcal{G}-\mathcal{F}^2}}\left(\mathcal{F}\frac{\partial X}{\partial v}-\mathcal{E}\frac{\partial X}{\partial u}\right).\end{aligned}\right\}\tag{52}$$

Hence we have the theorem:

THEOREM VI. *When the spherical representation of the asymptotic lines of a surface S satisfies the condition* (51), *one can find by quadratures two surfaces S_0 and S_{02}, associate to S, upon which the parametric lines form an isothermal-conjugate system, the parameters being such that*

$$D_0=D_0'',\qquad D_{02}=-D_{02}'';\tag{53}$$

and S_0 and S_{02} are associate surfaces of one another.

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* Bianchi, I, 166; German translation, p. 134; Eisenhart, p. 200.

The last statement follows from the fact that $D_0 D''_{02} + D''_0 D_{02} = 0$.*

9. In order to connect up this result with the results of the preceding sections, we remark that another form of equations (2) is:†

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= \frac{D'}{\sqrt{\mathcal{E}\mathcal{G}-\mathcal{F}^2}}\left(\mathcal{F}\frac{\partial X}{\partial u} - \mathcal{E}\frac{\partial X}{\partial v}\right), \\ \frac{\partial x}{\partial v} &= \frac{D'}{\sqrt{\mathcal{E}\mathcal{G}-\mathcal{F}^2}}\left(-\mathcal{G}\frac{\partial X}{\partial u} + \mathcal{F}\frac{\partial X}{\partial v}\right). \end{aligned}\right\} \tag{54}$$

From (52) and (54) it follows that

$$\frac{\partial x_0}{\partial u} = \frac{D_0}{D'}\frac{\partial x}{\partial v}, \quad \frac{\partial x_0}{\partial v} = \frac{D''_0}{D'}\frac{\partial x}{\partial u}. \tag{55}$$

From equations (8) and (2) we find that

$$\left.\begin{aligned} \frac{x_{0u}}{x_v} &= \frac{1}{F^2(\xi, \eta, \zeta)}\begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{1uu} \end{vmatrix}, \\ \frac{x_{0v}}{x_u} &= \frac{1}{F^2(\xi, \eta, \zeta)}\begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{1vv} \end{vmatrix}, \end{aligned}\right\} \tag{56}$$

where the subscripts u and v indicate differentiation with respect to these parameters, where σ of (8) has been replaced by σ_1, and where the omitted terms are similar to those given.

Hence a necessary and sufficient condition that S_0 be of the kind defined in THEOREM VI is that σ_1 satisfy the condition

$$\begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{1uu} \end{vmatrix} = \begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{1vv} \end{vmatrix}. \tag{57}$$

This equation may be written in the form

$$\begin{vmatrix} \xi_u\sigma_1 - \xi\sigma_{1u}, & \eta_u\sigma_1 - \eta\sigma_{1u}, & \zeta_u\sigma_1 - \zeta\sigma_{1u} \\ \xi_v\sigma_1 - \xi\sigma_{1v}, & \ldots\ldots\ldots, & \ldots\ldots\ldots \\ \xi_{uu}\sigma_1 - \xi\sigma_{1uu}, & \ldots\ldots\ldots, & \ldots\ldots\ldots \end{vmatrix} = \begin{vmatrix} \xi_u\sigma_1 - \xi\sigma_{1u}, & \ldots, & \ldots \\ \xi_v\sigma_1 - \xi\sigma_{1v}, & \ldots, & \ldots \\ \xi_{vv}\sigma_1 - \xi\sigma_{1vv}, & \ldots, & \ldots \end{vmatrix} \tag{58}$$

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* In the *Mathematische Annalen*, Vol. LXII (1906), pp. 524–529, we have given examples of surfaces which satisfy the requirements of THEOREM VI.

† Bianchi, I, 153; Eisenhart, p. 192.

which by means of (20) is equivalent to

$$\begin{vmatrix} \xi_1 & & & \\ & \eta_{1u} & & \\ & & \zeta_{1v} & \\ & & & \left(\frac{1}{\sigma_1}\right)_{uu} \end{vmatrix} = -\begin{vmatrix} \xi_1 & & & \\ & \eta_{1u} & & \\ & & \zeta_{1v} & \\ & & & \left(\frac{1}{\sigma_1}\right)_{vv} \end{vmatrix}. \tag{59}$$

This equation results also from the following theorem which we have established elsewhere:*

THEOREM VII. *The asymptotic lines on any one of a set of three surfaces S, S_1, S_0 correspond to a conjugate system on each of the other two surfaces.* For, from this theorem it follows that

$$D_0 D_1'' + D_0'' D_1 = 0, \quad D_1 D_3'' + D_1'' D_3 = 0, \quad D_3 D_7'' + D_3'' D_7 = 0.$$

Since $D_0 = D_0''$, we find that $D_7 = -D_7''$, and consequently from equations analogous to (55) and (56) we obtain (59).

In like manner we find that the function σ_2 which determines the surface S_{02}, defined in THEOREM VI, must satisfy the condition

$$\begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{2uu} \end{vmatrix} = -\begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{2vv} \end{vmatrix}. \tag{60}$$

Since the parametric curves on S_7 form an isothermal-conjugate system, there is a surface Σ associate to S_5 which is also associate to S_7, by THEOREM VI. We inquire whether the surface Σ is the surface S_7' of (41). On this hypothesis the equation obtained by replacing ξ, η, ζ, σ_1 in (58) by ξ_1, η_1, ζ_1, σ_1' respectively must hold. By means of (20) and (31) this equation is reducible to

$$\begin{vmatrix} \xi\sigma_1' - \xi_1\sigma_2, & \dots, & \dots, & \sigma_1 \\ \xi_u\sigma_1' - \xi_1\sigma_{2u}, & \dots, & \dots, & \sigma_{1u} \\ \xi_v\sigma_1' - \xi_1\sigma_{2v}, & \dots, & \dots, & \sigma_{1v} \\ \xi_{uu}\sigma_1' - \xi_1\sigma_{2uu}, & \dots, & \dots, & \sigma_{1uu} \end{vmatrix} = -\begin{vmatrix} \xi\sigma_1' - \xi_1\sigma_2, & \dots, & \dots, & \sigma_1 \\ \xi_u\sigma_1' - \xi_1\sigma_{2u}, & \dots, & \dots, & \sigma_{1u} \\ \xi_v\sigma_1' - \xi_1\sigma_{2v}, & \dots, & \dots, & \sigma_{1v} \\ \xi_{vv}\sigma_1' - \xi_1\sigma_{2vv}, & \dots, & \dots, & \sigma_{1vv} \end{vmatrix} \tag{61}$$

If equations (57) and (60) be written in the abbreviated form

$$\left.\begin{aligned} A(\sigma_{1uu} - \sigma_{1vv}) + (B - B_1)\sigma_{1u} + (C - C_1)\sigma_{1v} + (D - D_1)\sigma_1 = 0, \\ A(\sigma_{2uu} + \sigma_{2vv}) + (B + B_1)\sigma_{2u} + (C + C_1)\sigma_{2v} + (D + D_1)\sigma_2 = 0, \end{aligned}\right\} \tag{62}$$

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* *Differential Geometry*, p. 384.

we remark that $A, B, \ldots, D_1$ are such that ξ, η, ζ satisfy the equations

$$\begin{aligned} A\,\xi_{uu} + B\,\xi_u + C\,\xi_v + D\,\xi &= 0, \\ A\,\xi_{vv} + B_1\,\xi_u + C_1\,\xi_v + D_1\,\xi &= 0. \end{aligned}$$

By means of these quantities equation (61) can be reduced to

$$\begin{vmatrix} \xi\sigma_1' - \xi_1\sigma_2, & & & \sigma_1 \\ \xi_u\sigma_1' - \xi_1\sigma_{2u}, & & & \sigma_{1u} \\ \xi_v\sigma_1' - \xi_1\sigma_{2v}, & & & \sigma_{1v} \\ 0, & 0, & 0, & A(\sigma_{1uu} + \sigma_{1vv}) + (B + B_1)\sigma_{1u} \\ & & & + (C + C_1)\sigma_{1v} + (D + D_1)\sigma_1 \end{vmatrix} = 0. \quad (63)$$

From (56) it follows that if both members of (57) vanish, S_0 is a point. Excluding this case, equation (63) reduces to

$$\begin{vmatrix} \xi_1 \cdots\cdots \sigma_1' \\ \xi \cdots\cdots \sigma_2 \\ \xi_u \cdots\cdots \sigma_{2u} \\ \xi_v \cdots\cdots \sigma_{2v} \end{vmatrix} = 0.$$

If we put

$$\xi_1 = a\xi + b\xi_u + c\xi_v, \quad (64)$$

where a, b, c are functions of u and v, the first of equations (20) is reducible to

$$\xi\left[(a-1)\frac{\sigma_{1u}}{\sigma_1} + a_u + cM\right] + \xi_u\left(1 + a + \frac{\sigma_{1u}}{\sigma_1}b + b_u\right) + \xi_v\left(c\frac{\sigma_{1u}}{\sigma_1} + c_u\right) + b\xi_{uu} = 0. \quad (65)$$

Since this equation is satisfied also by η, ζ and σ_2, it follows that unless the right-hand member of equation (60) is zero, all the coefficients in (65) are equal to zero. Similar results follow when the value (64) of ξ_1 is substituted in the second of (20). Combining these results we are led to the conclusion that S_{02} must be a point. Hence if equation (61) is to be satisfied, either S_0 or S_{02} must be a point. When S_0 is a point, both S_1 and S_3 are planes, and S_7 is a point.

In a similar manner it can be shown that the parameters of the parametric conjugate system on S_{72}' are isothermal-conjugate only when S_0 or S_{02} is a point.

In consequence of THEOREM VII the foregoing results may be stated thus:

THEOREM VIII. *If the conjugate parametric system on S_0 is such that $D_0 = D_0''$ and if S_{02} is a point, the conjugate parametric system on each of the surfaces* (41), *except S, S_5, S_{52}, S', is such that $D_i = D_i''$ or $D_i = -D_i''$; the pairs of surfaces S_0'', S_{02}''; S_7, S_7'' are associate surfaces; and this is the only case in which a group of surfaces* (41) *possesses these properties.*

10. We proceed to an investigation of the problem as to the coincidence of certain surfaces of the group (41), when S_0 and S_{02} satisfy the requirements of THEOREM VI, and neither is a point.

From the results of § 5 we know that S_0 cannot coincide with S_1, S_3, S_5 or S_7. Since conditions (53) are satisfied, it follows from THEOREM VII that

$$D_{12} = D_{12}'', \quad D_{32} = -D_{32}''.$$

Hence S_0 cannot coincide with S_{32}, but it remains for us to see whether S_0 and S_{12} can coincide.

On this hypothesis we have

$$\Sigma X \frac{\partial x_{12}}{\partial u} = 0, \quad \Sigma X \frac{\partial x_{12}}{\partial v} = 0. \tag{66}$$

It is known that in general*

$$\Sigma X \frac{\partial x_{12}}{\partial u} = \frac{D' \frac{\partial \phi}{\partial u} - D \frac{\partial \phi}{\partial v}}{K\sqrt{EG - F^2}}, \quad \Sigma X \frac{\partial x_{12}}{\partial v} = \frac{D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v}}{K\sqrt{EG - F^2}},$$

where ϕ is a solution of the equation

$$\frac{1}{\sqrt{EG - F^2}} \left\{ \frac{\partial}{\partial u}\left(\frac{D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v}}{K\sqrt{EG - F^2}} \right) + \frac{\partial}{\partial v}\left(\frac{D \frac{\partial \phi}{\partial v} - D' \frac{\partial \phi}{\partial u}}{K\sqrt{EG - F^2}} \right) \right\}$$
$$= \frac{2FD' - ED'' - GD}{EG - F^2} \phi. \tag{67}$$

In order that (66) be satisfied, ϕ must be a constant and consequently S must be a minimal surface, as is seen from (67). Hence an adjoint of a minimal surface may be taken for both S_0 and S_{12}.† In this case S_0 is a surface of negative curvature. Consequently in order that the conditions (53) may be

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[*] Bianchi, II, 6; German translation, p. 290; Eisenhart, p. 375.

[†] Cf. *American Journal, loc. cit.*, p. 196.

preserved, we take the adjoint of S for S_{02} and S_1. When now we compare equations (7) with the formulas of Schwarz,[*] namely

$$\begin{aligned} dx_1 &= Ydz - Zdy, \\ dy_1 &= Zdx - Xdz, \\ dz_1 &= Xdy - Ydx, \end{aligned}$$

we notice that S_0 is the unit sphere. Since S is adjoint to S_1, we have

$$dx = - Ydz_1 + Zdy_1,$$

and similar expressions in y and z. Hence S_3 is the unit sphere also, but corresponding points of S_0 and S_3 are diametrically opposite. Since S_5 corresponds to the sphere S_3 with orthogonality of linear elements, S_5 is the mean surface of an isotropic congruence.[†] In consequence of THEOREM VIII neither S_0 nor S_{02} can coincide with any members of the group (41) other than those which we have considered.

We suppose now that S_0 is such that the first of equations (53) holds, but that the second of these equations is not satisfied by S_{02}. If S_0 and S''_3 were to coincide, S_7 and S''_7 would be such that $D_{7'} = - D''_{7'}$, $D_7 = - D''_7$, which is impossible. If S_0 and S''_1 were to coincide, we should have $D_{7'} = D''_{7'}$, as well as $D_7 = - D''_7$, and consequently this is the same problem as determining whether S_7 and S_{32} can coincide, when both of the conditions (53) are satisfied. If S_7 and S_{32} coincide, so also do their polar reciprocals S_1 and S_{02}. But we have seen that this is true only when S is minimal, and S_1 and S_{02} are its adjoint.

It remains for us to determine under what conditions S_0 can coincide with S''_{12} or S''_{32}. From (42) it follows that if S_0 and S''_{12} coincide

$$\xi_2 = x_0 \sigma'_2, \quad \eta_2 = y_0 \sigma'_2, \quad \zeta_2 = z_0 \sigma'_2.$$

When these values are substituted in equation (35) and it is recalled that σ'_2 also is a solution of this equation, we find that x_0, y_0, z_0 are solutions of

$$\frac{\partial^2 \theta}{\partial u \, \partial v} + \frac{\partial \log \sigma'_2}{\partial v} \frac{\partial \theta}{\partial u} + \frac{\partial \log \sigma'_2}{\partial u} \frac{\partial \theta}{\partial v} = 0.$$

By means of (49) the point equation of S_0 is reducible to

$$\frac{\partial^2 \theta}{\partial u \, \partial v} + \varepsilon \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' \frac{\partial \theta}{\partial u} + \varepsilon \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}' \frac{\partial \theta}{\partial v} = 0,$$

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[*] Bianchi, Vol. II, p. 327; German translation, p. 377; Eisenhart, p. 265.

[†] Darboux, IV, p. 95. In the subsequent pages Darboux determines all the surfaces of the group represented in Fig. 2, when S is a sphere.

where ε is ± 1 according as the curvature of S_0 is positive or negative. Hence σ_2' is given by the quadratures

$$\frac{\partial \log \sigma_2'}{\partial u} = \varepsilon \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}', \quad \frac{\partial \log \sigma_2'}{\partial v} = \varepsilon \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}'. \tag{68}$$

We have shown elsewhere* that when the condition (51) is satisfied, the surface S_0 is given by the quadratures

$$\frac{\partial x_0}{\partial u} = \mu \frac{\partial x}{\partial v}, \quad \frac{\partial x_0}{\partial v} = \varepsilon \mu \frac{\partial x}{\partial u},$$

where, according as ε is ± 1, S_0 is of positive or negative curvature, and μ is determined by

$$\frac{\partial \log \mu}{\partial v} = \varepsilon \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}, \quad \frac{\partial \log \mu}{\partial u} = \varepsilon \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}, \tag{69}$$

the Christoffel symbols being formed with respect to the linear element of S. If K denotes the total curvature of S and ρ is defined by

$$\frac{1}{\rho^2} = -K,$$

it is possible to derive from (68) and (69) the relation

$$\mu \sigma_2' = \rho^{\frac{3}{2}} \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}, \tag{70}$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ have the same significance as in (48). Hence the determination of S_0 carries with it that of σ_2'. Likewise, when S_0 is known, the corresponding function σ_1 is found directly from (11), (13) and

$$\Sigma x_3 \frac{\partial x_0}{\partial u} = 0, \quad \Sigma x_3 \frac{\partial x_0}{\partial v} = 0.$$

In the next place σ_2 must be found, and from (32) it follows that this problem reduces to the quadratures

$$\frac{\partial \log \sigma_2}{\partial u} = -\frac{\dfrac{\partial \sigma_2'}{\partial u} + \dfrac{\partial \sigma_1}{\partial u}}{\sigma_2' - \sigma_1}, \quad \frac{\partial \log \sigma_2}{\partial v} = -\frac{\dfrac{\partial \sigma_2'}{\partial v} - \dfrac{\partial \sigma_1}{\partial v}}{\sigma_2' + \sigma_1}. \tag{71}$$

The condition of integrability of these equations is reducible to

$$\left(\frac{\partial^2 \sigma_2'}{\partial u \partial v} + M \sigma_2'\right)(\sigma_2' + \sigma_1)(\sigma_2' - \sigma_1) - 2\sigma_2' \left(\frac{\partial \sigma_2'}{\partial v} - \frac{\partial \sigma_1}{\partial v}\right)\left(\frac{\partial \sigma_2'}{\partial u} + \frac{\partial \sigma_1}{\partial u}\right) = 0.$$

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* *Mathematische Annalen*, Vol. LXII (1906), pp. 507, 520.

Moreover, it is readily shown that when this condition is satisfied, the function σ_2 given by (71) is a solution of equation (1). The solution of our problem depends therefore upon the determination of the surfaces S whose asymptotic lines satisfy the condition (51). As we have been unable to get the general solution of this problem,* we cannot determine the conditions under which S_0 and S'_{12} coincide. It should be remarked that from THEOREM VII it follows that if the conditions of the problem were satisfied the surfaces S_5, S_{52} and S' also would satisfy the condition (51).

The problem of the coincidence of S_0 and S'_{32} is similar to the preceding one, the difference being that equation (70) must be replaced by

$$\frac{\mu}{\sigma'_2} = \rho^{\frac{3}{2}} \sqrt{\mathcal{E}\mathcal{G}-\mathcal{F}^2}.$$

Having thus considered the possibilities of S_0 conciding with other surfaces of the group (41), it is unnecessary to go any further. For, the only other type is S_1 and its polar reciprocal S_7 is of the same type as S_0.

PRINCETON, *April* 20, 1909.

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[*] *Mathematische Annalen*, Vol. LXII (1906), pp. 534–536. Here we have shown that minimal surfaces and pseudospherical surfaces of revolution satisfy the condition (51).